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Conformational Energy of the 5-Membered Ring. Implication of Geometric and Energetic Properties in the Conformational Characteristics

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Abstract. In a previous article (8) a geometrical study of the five-membered ring showed that: a) for the case of the 20 symmetrical C_2 and C_S conformations, the pseudorotation formulae for the torsion angles are a geometrical property of the ring; b) geometrical considerations alone are unable to define the puckering amplitude, the bond angle values, and the pathway between two symmetrical conformations. Here we examine how the energy equations enable us to define the deformation amplitude χ_m, establish the bond angles expressions and check the energy invariability along the pseudorotation circuit. The problem is next developed fully in the case where the bond and torsional energy only are considered: the literal expression[1] of χ_m is then given as a function of the bond angle ω which cancels out the bond angle energy. A numerical application is carried out on cyclopentane and the values of the parameters K^t, K^1 and ω used in the conformational energy calculations are considered.

Key words: Five-membered ring – Pseudorotation – Conformational analysis – Molecular geometry

Introduction

Cyclic structures are very often encountered in molecules of biological interest. The conformation of the ring, governing the orientations of groups fixed on the component atoms, is therefore implicated in conformation-activity relations. If this conformation is well understood substituents can be chosen and set in place for the purpose of changing the biological activity. To achieve such an understanding, as pointed out by Altona in 1971 [1], it is necessary to evaluate the energetic and geometrical factors involved.

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[1] In the literal expressions the angles are expressed in radians. The energies are expressed in kcal/mole, K^t in Kcal/mole, K^1 en kcal/mole rad^2
The IUPAC conventions are used (14)

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Many experimental and theoretical studies have led to the concept of pseudo-rotation. In the case of cyclopentane this idea was advanced in 1947 by Pitzer et al. [2], who concluded from combined thermodynamic and spectroscopic results that the non-planar ring exists in a mixture of interconverting C_2 and C_S forms; according to these authors' hypothesis the displacement z_j of the j^{th} atom perpendicularly to the planar ring takes the form (using their own notations):

$$z_j = \sqrt{\frac{2}{5}}\, q \cos 2(\phi + 2\pi j/5)$$

with q the amplitude of the movement and ϕ the phase angle which on varying allows the supposedly isoenergetic passage from one form to another. Then in 1960 Hendrickson [3] calculated the conformational energies of cyclopentane for different torsional and bond angle values. Taking conformations of equal and minimal energy he observes that they contain the C_2 and C_S conformations belonging to the pseudo-rotation circuit and supposes them to include the whole set of conformations belonging to the entire circuit.

In 1967 Geise et al. [4] combined these two approaches and making use of Hendrickson's energy calculation results found that in the constant energy region the torsional angles are expressed by the relationship (using their own notations):

$$\phi_j = a \sin(P + 4\pi j/5)\,,$$

where a is the amplitude of the vibrational angular motion; these authors also found that in the same region the bond angles take the form:

$$\nu_j = \tau + b \sin[2(P + 4\pi j/5) + \delta]$$

and therefore vary between the values $\tau - b$ and $\tau + b$.

The pseudo-rotation concept thus has two aspects, proceeding:

experimentally from the hypothesis of vibration of atoms on either side of the planar ring;

theoretically from an energy calculation, the minimal energy conformations being those for which the torsional and bond angles obey the pseudorotational equations for a given deformation amplitude.

In 1968 Lifson and Warshel [5] once more confirmed the existence of the pseudo-rotational path, but because of the difference in the potential functions used for the calculation believe that this equipotential path is due more to the geometry of cyclopentane than to the exact nature of the molecular interactions.

In 1971 Altona [1] pointed out that further progress in the conformational analysis of 5-membered rings will proceed from a better understanding of the energetic and geometric factors involved.

In 1972, Dunitz [6], founding upon the Pitzer's [2] relations provided a mathematical derivation of the pseudorational formulae for the torsional angles.

No complete mathematical demonstration of the pseudo-rotational formulae having been given [7] we have tried this approach with a view to differentiating geometrical from energetic properties.

In a previous article [8] we showed that in a regular 5-membered ring, where the angular variations were limited, the approximate pseudo-rotation formulae $\chi_i = \chi_m \cos[P + (i-1)\delta]$ relative to the torsional angles were due for a given puckering amplitude to geometrical limitations alone and could hence be considered independent of any vibrational or energetic hypotheses.

The present article deals with energetic considerations.

The notations used are given in the appendix, together with the units whenever necessary.

I. General Energetic Considerations

The conformation assumed by a molecule is that which corresponds to its energy minimum. This conformational energy must therefore be expressed as a function of the bond parameters 1_i, ϕ_i and the torsion parameter χ_i while the minimal energy condition $dE = 0$, will provide as many relationships as geometrically independent parameters. For this purpose geometrical relations were previously stated [8].

Conformational Energy Expression

In the empirical approach this energy is broken down into several terms [9]:
Energy of interaction between the non-bound atoms i and j:
Van der Waals energy which takes the Lennard-Jones form:

$$E_{ij}^{VdW} = A_{ij}/r_{ij}^{12} - B_{ij}/r_{ij}^{6} .$$

Electrostatic energy:

$$E_{ij}^{él} = q_i q_j / e r_{ij} .$$

Hydrogen bond energy:

$$E_{ij}^{LH} = D_{ij}/r_{ij}^{12} - C_{ij}/r_{ij}^{6} + S(r_{ij}) .$$

Bond angle energy:

$$E_i^{1} = K_i^{1} (\phi_i - \omega_i)^2 .$$

Torsional energy:

$$E_i^{t} = \frac{1}{2} K_i^{t} (1 + \cos 3\chi_i) \qquad \text{for the } C{-}C \text{ bonds.}$$

The bond elongation energy is not accounted for: the bond lengths are considered to be constant.

The conformational energy of the molecule is obtained by summing these components over the pairs of atoms, bonds and bond angles making up this molecule:

$$E = \sum_{\substack{\text{atom}\\ i}} \sum_{\substack{\text{atom}\\ j \neq i}} (E_{ij}^{VdW} + E_{ij}^{él} + E_{ij}^{LH}) + \sum_{\substack{\text{bond}\\ k}} E_k^{t} + \sum_{\substack{\text{bond}\\ \text{angle } i}} E_i^{1} .$$

Each of these terms involves not only geometrical characteristics (inter-atomic distances r_{ij}, torsional angles χ_i, bond angles ϕ_i) but also parameters dependent on the nature of the atoms or atom pairs concerned (torsional and bond constant: K_i^t *and* K_i^l, electric charges q_i, bond angles ω_i corresponding to zero bond energy, coefficients A_{ij}, B_{ij}, C_{ij}, D_{ij}). The general case of any ring is therefore difficult to handle, and this work will be confined to the case of the regular cyclopentane type ring.

I.1 Conformational Energy of the Regular Cyclopentane Type Ring

The different energy components involve the bond angles ϕ_i, the torsional angles χ_i and the inter-atomic distances r_{ij}, these last being expressed as a function of the bond length 1 and the trigonometric functions of the angles ϕ_i and χ_i. These functions are developed in powers of ε_i and χ_i and the ring deformations being limited, we stopped these expansions at the power two [8]. (We write $\phi_i = \phi + \varepsilon_i$, the ϕ value $= 3\pi/5$ corresponding to the no puckered ring).

The general expression of the conformational energy is thus written:

$$E = A + B\sum_i \varepsilon_i + C\sum_i \chi_i + D\sum_i \varepsilon_i^2 + F\sum_i \chi_i^2 .$$

Application of the previously established approximate geometrical relations [6, 8]:

$$\sum_i \chi_i = 0 \quad \text{and} \quad \sum_i \chi_i^2 = k\sum_i \varepsilon_i$$

reduces the first terms of this development to the following expression:

$$\boxed{E = A_0 + A_1\sum_i \varepsilon_i + A_2\sum_i \varepsilon_i^2} \; . \tag{1}$$

The properties set forth from now on have been established within the limit of validity of this development in which the crossterms are not considered.

Any deformation of the ring such that $\sum_i \varepsilon_i =$ *constant [preserving [8] the puckering amplitude] and* $\sum_i \varepsilon_i^2 =$ *constant will thus take place at constant energy.*

I.2 Expression of the Energy of a Regular Ring in Symmetrical Conformations

For the two kinds of symmetry we have:

$$\varepsilon_1 = \varepsilon_2$$

and

$$\varepsilon_3 = \varepsilon_5 .$$

Let us pose $\varepsilon_T = \sum_i \varepsilon_i$

$$\varepsilon_T = 2\,\varepsilon_1 + 2\,\varepsilon_3 + \varepsilon_4$$

and $\sum_i \varepsilon_i^2 = 2\,\varepsilon_1^2 + 2\,\varepsilon_3^2 + \varepsilon_4^2\,.$

I.2a C$_S$ Symmetry. In the case of the C_S symmetry we know [8] that:

$$\varepsilon_4 = -4\,\varepsilon_1 \cos 2\phi\,.$$

The expressions of $\sum_i \varepsilon_i$ and $\sum_i \varepsilon_i^2$ become:

$$\varepsilon_T = 2\,\varepsilon_3 + 2\,\varepsilon_1(1-2\cos 2\phi)$$

and

$$\sum_i \varepsilon_i^2 = 2\,\varepsilon_3^2 + 2\,\varepsilon_1^2\,(3-4\cos 2\phi)\,.$$

Whence the expression of energy E as a function of the 2 geometrically independent variables ε_1 and ε_3:

$$\begin{aligned} E(\varepsilon_1, \varepsilon_3) = A_0 &+ 2A_1[\varepsilon_1(1-2\cos 2\phi) + \varepsilon_3] \\ &+ 2A_2[\varepsilon_1^2(3-4\cos 2\phi) + \varepsilon_3^2] \end{aligned}$$

If we express this energy as a function of the 2 variables ε_1 and ε_T, also independent, we get:

$$2\,\varepsilon_3 = \varepsilon_T - 8\,\varepsilon_1 \cos^2 2\phi$$

$$\sum_i \varepsilon_i^2 = 40\,\varepsilon_1^2 \cos^2 2\phi + \varepsilon_T^2/2 - 8\,\varepsilon_1\varepsilon_T \cos^2 2\phi$$

i.e. $$\boxed{E_{\mathrm{CS}}(\varepsilon_1, \varepsilon_T) = A_0 + A_1\varepsilon_T + A_2\left[40\,\varepsilon_1^2 \cos^2 2\phi + \varepsilon_T^2/2 - 8\,\varepsilon_1\varepsilon_T \cos^2 2\phi\right]}\,. \quad (2)$$

I.2b C$_2$ Symmetry. In the case of the C$_2$ symmetry we know [8] that:

$$\varepsilon_4 = -4\,\varepsilon_1 \cos 2\phi - 2\,\varepsilon_T/\sqrt{5}\,.$$

Hence

$$\varepsilon_3 = \varepsilon_T(2 + \sqrt{5})/2\,\sqrt{5} + \varepsilon_1(2\cos 2\phi - 1)$$

and

$$\sum_i \varepsilon_i^2 = 40\,\varepsilon_1^2 \cos^2 2\phi + (\varepsilon_T^2/10)(13-16\cos 2\phi) - 24\,\varepsilon_1\varepsilon_T \cos^2 2\phi\,.$$

The expression of energy E as a function of the two geometrically independent variables ε_1 and ε_T is:

$$\boxed{\begin{aligned} E_{C2}(\varepsilon_1, \varepsilon_T) = A_0 + A_1\varepsilon_T + A_2[40\varepsilon_1^2 \cos^2 2\phi \\ + (\varepsilon_T^2/10)(13-16\cos 2\phi) - 24\varepsilon_1\varepsilon_T \cos^2 2\phi] \end{aligned}} \ . \tag{3}$$

II.1 *Characteristics of Minimal Energy C_S and C_2 Conformations*

The ε_1 and ε_T values corresponding to minimal conformational energy will be obtained by writing:

$$\frac{dE}{d\varepsilon_1} = \frac{dE}{d\varepsilon_T} = 0 \ .$$

II.1a *C_S Symmetry*

$$1) \ \frac{dE}{d\varepsilon_1} = 0 \quad \text{leads to:} \quad 10\varepsilon_1 = \varepsilon_T.$$

From which we deduce the set of ε_i values:

$$\boxed{\begin{aligned} \varepsilon_1 = \varepsilon_2 &= \varepsilon_T/10 \ , \\ \varepsilon_3 = \varepsilon_5 &= \varepsilon_T(2 + \cos 2\phi)/5 \ , \\ \varepsilon_4 &= -2\varepsilon_T(\cos 2\phi)/5 \ . \end{aligned}} \qquad \begin{aligned} (4) \\ (5) \\ (6) \end{aligned}$$

In symmetrical C_S conformations the ratios $\varepsilon_i/\varepsilon_T$ are constant and independent of the formulation used for the interaction energy calculation and of the value of the parameters involved.

$$2) \ \frac{dE}{d\varepsilon_T} = 0 \quad \text{leads to:} \quad A_1 + A_2\varepsilon_T + 2A_2\varepsilon_I(2\cos 2\phi - 1) = 0$$

$$\text{i.e.:} \quad \varepsilon_T - 8\varepsilon_1 \cos^2 2\phi = -A_1/A_2 \ . \tag{7}$$

Resolving the system of Eqs. (4) and (7) we get:

$$\boxed{\varepsilon_T = -(5/2)(A_1/A_2)/(2 + \cos 2\phi)} \ . \tag{8}$$

The optimal deformation value is proportional to the ratio A_1/A_2.

Knowing the relationship [8] between χ_m and ε_T:

$$\chi_m^2 \sin^2\phi \sin 2\phi = \varepsilon_T \quad \text{i.e.} \quad \sqrt{5}\,\chi_m^2 \sin\phi = -4\varepsilon_T$$

we obtain the expression of χ_m:

$$\boxed{\chi_m^2 = \frac{2\sqrt{5}\,A_1/A_2}{\sin\phi\,(2+\cos 2\phi)}}\ .$$

ε_T being given by relation (8), the ε_i values in type C_S conformations become:

$$\begin{cases} \varepsilon_1 = \varepsilon_2 = -(A_1/4A_2)/(2+\cos 2\phi) \\ \varepsilon_3 = \varepsilon_5 = -(A_1/2A_2) \\ \quad \varepsilon_4 = (A_1/A_2)\cos 2\phi/(2+\cos 2\phi) \end{cases}$$

and the energy value:

$$E_{CS} = A_0 + A_1\varepsilon_T + A_2(2+\cos 2\phi)\varepsilon_T^2/5$$

$$E_{CS} = A_0 - (5/4)(A_1^2/A_2)/(2+\cos 2\phi) = A_0 + A_1\varepsilon_T/2\ . \tag{9}$$

A_0 being the energy corresponding to the planar ring ($\varepsilon_T = 0$) the decrease in energy due to deformation of the ring is:

$$\boxed{\Delta E = \frac{5A_1^2}{4A_2(2+\cos 2\phi)} = -\frac{A_1}{2}\varepsilon_T}\ . \tag{10}$$

II.Ib. C_2 Symmetry

1) $\dfrac{dE}{d\varepsilon_1} = 0$ leads to: $10\,\varepsilon_1 = 3\,\varepsilon_T$

from which we deduce the set of ε_i values:

$$\boxed{\begin{array}{l} \varepsilon_1 = \varepsilon_2 = 3\,\varepsilon_T/10 \\ \varepsilon_3 = \varepsilon_5 = -(\varepsilon_T/5)\cos 2\phi \\ \quad \varepsilon_4 = (2\,\varepsilon_T/5)\,(1+\cos 2\phi) \end{array}}\ . \tag{11}$$

In symmetrical C_2 type conformations, as in the C_S one's the ratios $\varepsilon_i/\varepsilon_T$ are constant and independent of the formulation used for the interaction energy calculation and of the value of the parameters involved.

2) $\dfrac{dE}{d\varepsilon_T} = 0$ leads to:

$$(\varepsilon_T/5)(13-16\cos 2\phi) - 24\varepsilon_1\cos^2 2\phi = -A_1/A_2$$

with $\varepsilon_1 = 3\varepsilon_T/10$, we obtain:

$$\boxed{\varepsilon_T = -(5/2)(A_1/A_2)/(2+\cos 2\phi) = -(5/2)(A_1/A_2)(7+\sqrt{5})/11}\ .$$

The optimal puckering value is the same for the C_2 conformation as for the C_S one. It is proportional to the ratio A_1/A_2.

Putting this ε_T value in the ε_i (11) expressions

$$\begin{cases} \varepsilon_1 = \varepsilon_2 = -(3/4)(A_1/A_2)/(2+\cos 2\phi) = -(3/4)(A_1/A_2)(7+\sqrt{5})/11 \\ \varepsilon_3 = \varepsilon_5 = (1/2)(A_1/A_2)\cos 2\phi/(2+\cos 2\phi) = -(1/2)(A_1/A_2)(3+2\sqrt{5})/11 \\ \varepsilon_4 = -(A_1/A_2)(1+\cos 2\phi)/(2+\cos 2\phi) = -(A_1/A_2)(4-\sqrt{5})/11 \end{cases}$$

and the energy value becomes:

$$E_{C2} = A_0 + A_1\varepsilon_T + A_2\varepsilon_T^2(2+\cos 2\phi)/5 \, .$$

The optimal C_2 and C_S conformations have the same conformational energy:

$$E_{C2} = E_{CS} = A_0 + A_1\frac{\varepsilon_T}{2} = A_0 - \frac{5A_1^2}{4A_2(2+\cos 2\phi)} = A_0 - \frac{5A_1^2(7+\sqrt{5})}{4A_2(11)}$$

and for the two kinds of conformation the decrease in energy due to deformation of the ring is:

$$\Delta E = -\frac{A_1}{2}\varepsilon_T = -\frac{A_1}{2}\chi_m^2\sin^2\phi\sin 2\phi = \frac{5A_1^2}{4A_2(2+\cos 2\phi)} \, .$$

II.2 Pseudorotation Formulae Relative to the Bond Angles

The torsional angles have been shown to obey the approximate equations:

$$\chi_i = \chi_m \cos\,[\mathrm{P} + (i-1)\delta]$$

for solely geometrical reasons [8]. We shall now show that the geometrically intermediate bond angles, once fixed by the condition $dE = 0$, satisfy expressions of the same type.

The relations established in [8] between torsional and bond angles being of the form:

$$\chi_j^2 = \sum_i k_i\varepsilon_i \, .$$

Hence: $\chi_m^2\cos^2[P+(j-1)\delta] = (\chi_m^2/2)\,\{1+\cos[2P+2(j-1)\delta]\} = \sum_i k_i\varepsilon_i$

we look for the ε_i expressions in the form:

$$\varepsilon_i = A + B\cos\{2[P+(i-1)\delta]+\gamma\} \, .$$

To obtain the C_S conformation in which $\chi_1 = 0$ we must have $P = \pi/2$. To then get $\varepsilon_1 = \varepsilon_2$ and $\varepsilon_3 = \varepsilon_5$ we need $\gamma = \pi/5$.
We then have:

$$\varepsilon_i = A - B\cos\,[2(i-1)\delta + \pi/5] \, .$$

Identification of these relations with the ε_i expressions (4, 5, 6) gives:

$$A = \varepsilon_T/5 \tag{12}$$

and

$$B = (2\varepsilon_T/5)\cos 2\delta \tag{13}$$

and

$$\boxed{\varepsilon_i = \frac{\varepsilon_T}{5} + \frac{2\varepsilon_T}{5}\cos 2\delta \cos\{2[P + (i-1)\delta] + \pi/5\}}\ . \tag{14}$$

The ε_i values thus oscillate between $(\varepsilon_T/5)(1 + 2\cos 2\delta)$ and $(\varepsilon_T/5)(1-2\cos 2\delta)$ around the mean value $\varepsilon_T/5$.

Using Hendrickson's numerical results [3], Geise et al. [4] found similar forms for the bond angles:

$$\nu_j = \tau + b\sin[2(P + 4\pi j/5) - 3\pi/10]$$

(keeping in this formula their own notations).

These authors found that the mean value τ and the amplitude b cannot be chosen independently. Proof of this impossibility is supplied here from the analytical expressions of these two quantities:

$$\tau = \phi + \varepsilon_T/5$$

and

$$b = 2(\varepsilon_T/5)\cos 2\delta = -2(\varepsilon_T/5)\cos\phi\ .$$

These two quantities are thus indeed related by the expression:

$$b = 2(\phi - \tau)\cos\phi\ .$$

The same authors also plotted τ and b versus the torsion angle variation amplitude a which we have called χ_m. We can give the literal expressions of these representations; knowing in fact the relationship between χ_m and ε_T:

$$\sqrt{5}\,\chi_m^2\sin\phi = -4\varepsilon_T\,,$$

we obtain:

$$\tau = \phi - \frac{\chi_m^2\sin\phi}{4\sqrt{5}} \quad\text{and}\quad b = \frac{\chi_m^2\sin 2\phi}{4\sqrt{5}}\ .$$

The mean value $\bar{\varepsilon}_i = \tau - \phi$ of the bond angles deformations and the amplitude of their variations b are proportional to χ_m^2.

We find again the parabolic forms given by Geise et al. [4] and the values at the origin (for $\chi_m = 0$: $\tau = \phi = 108°$ and $b = 0$).

III. Conformational Energy Variation During Pseudorotation

With the variation of angle P the ring undergoes a continuous deformation, passing through the 20 symmetrical C_S and C_2 conformations.

The question then is whether we meet with energy barriers or whether the energy remains constant along this circuit.

Let us express this energy from the ε_i relations established previously: These expressions take the form:

$$\begin{aligned}\varepsilon_i &= A + B\cos(P' + 2i\delta) \quad \text{with} \quad P' = 2P - 2\delta + \pi/5\\ \varepsilon_i^2 &= A^2 + B^2\cos^2(P' + 2i\delta) + 2AB\cos(P' + 2i\delta)\\ &= A^2 + \frac{B^2}{2} + \frac{B^2}{2}\cos(2P' + 4i\delta) + 2AB\cos(P' + 2i\delta)\end{aligned}$$

and

$$\sum_i \varepsilon_i^2 = 5A^2 + 5\frac{B^2}{2} + \frac{B^2}{2}\sum_i \cos(2P' + 4i\delta) + 2AB\sum_i \cos(P' + 2i\delta)$$

now,

$$\sum_i \cos(2P' + 4i\delta) = \sum_i \cos(P' + 2i\delta) = 0$$

whence

$$\sum_i \varepsilon_i^2 = 5A^2 + 5B^2/2\,.$$

Replacing A and B by their expressions (12) and (13):

$$\sum_i \varepsilon_i^2 = \frac{\varepsilon_T^2}{5}(1 + 2\cos^2 2\delta)\,. \tag{15}$$

Whence the energy equation:

$$E = A_0 + A_1\,\varepsilon_T + A_2\frac{\varepsilon_T^2}{5}(1 + 2\cos^2 2\delta)\,. \tag{16}$$

For a given puckering amplitude: ε_T, the energy, independent of the phase angle P, remains constant all along the pseudorotation circuit.

In particular, for the amplitude ε_T [8] corresponding to the minimal energy:

$$\boxed{E = A_0 - \frac{5A_1^2}{4A_2}\left(\frac{1}{2 + \cos\delta}\right)}$$

already found for the C_S conformation (let us recall [8] the relation $\cos 2\phi = \cos\delta$).

The results obtained so far are valid however the interaction energies are represented.

As Lifson and Warshel rightly anticipated [5], the existence of an isoenergic pseudorotational path is thus independent of the exact nature of the atomic interactions and derives from the fact that, following the geometrical properties of the ring, the conformational energy can be approximated by

$$\mathrm{E} - \mathrm{A}_0 + A_1 \sum_i (\phi_i - \phi) + A_2 \sum_i (\phi_i - \phi)^2 .$$

According to the different forms of interaction chosen the coefficients A_i of this development will vary; only the ratio A_1/A_2 will intervene in the definition of the optimal deformation χ_m, and only the ratio A_1^2/A_2 in that of the decrease in energy ΔE obtained on passing from the planar to the deformed ring.

The above results will be applied to the case where only the competition between bond and torsional energies is accounted for in the establishment of the optimal conformation.

IV. Application to the Case where only the Bond and Torsional Energies are Considered

The bond and torsional energies respectively are as follows:

$$E^1 = K^1 \sum_i (\phi_i - \omega)^2 = K^1 \sum_i (\phi_i - \phi + \phi - \omega)^2$$

$$\boxed{E^1 = 5K^1(\phi - \omega)^2 + 2K^1(\phi - \omega) \sum_i \varepsilon_i + K^1 \sum_i \varepsilon_i^2} \qquad (17)$$

$$E^t = (K^t/2) \sum_i (1 + \cos 3\chi_i) = 5K^t/2 + (K^t/2) \sum_i \cos 3\chi_i$$

$$= 5K^t/2 + (K^t/2) \sum_i \cos\chi_i \,(4\cos^2\chi_i - 3) .$$

Replacing each $\cos\chi_i$ by its development versus ε_i [8] and neglecting the $\varepsilon_i\,\varepsilon_j$ terms which contribute little compared with those in ε_i:

$$E^t = 5K^t - \frac{9K^t}{2}\left[\frac{4\cos 2\phi + 1}{\sin\phi}\right] \sum_i \varepsilon_i$$

i.e.: $$\boxed{E^t = 5K^t - 18K^t \sin 2\phi \sum_i \varepsilon_i .} \qquad (18)$$

Whence the total energy expression:

$$E = E_0 + 5K^1(\phi - \omega)^2 + 5K^t + [2K^1(\phi - \omega) - 18K^t \sin 2\phi] \sum_{\mathrm{i}} \varepsilon_i$$
$$+ K^1 \sum_{\mathrm{i}} \varepsilon_i^2 \qquad (19)$$

E_0 consisting of all the energy terms not defined in this expression and considered as constant during deformation of the ring.

Let us take: $$\begin{cases} A^0 = E_0 + 5K^1(\phi - \omega)^2 + 5K^t \\ A_1 = 2K^1(\phi - \omega) - 18K^t \sin 2\phi \\ A_2 = K^1 . \end{cases}$$

Expression [8] for ε_T becomes:

$$\boxed{\varepsilon_T = \frac{5}{2 + \cos 2\phi}\left[(\omega - \phi) + 9\,\frac{K^t}{K^1}\sin 2\phi\right]} \tag{20}$$

and the decrease in energy (10) ΔE due to deformation of the ring:

$$\boxed{\Delta E = \frac{A_1}{2}\,\varepsilon_T = \frac{5}{2 + \cos 2\phi}\,K^1\left[(\omega - \phi) + 9\,\frac{K^t}{K^1}\sin 2\phi\right]^2} \,. \tag{21}$$

Knowing [8] that $\chi_{\mathrm{m}}^2 = -\,(4/\sqrt{5})\ \varepsilon_T/\sin\phi = -\ 1.88\ \varepsilon_T$, the literal expression of χ_{m}^2 as a function of ω and of the ratio K^t/K^1 is written:

$$\boxed{\chi_{\mathrm{m}}^2 = -\,\frac{4\sqrt{5}\,[(\omega - \phi) + 9\,(K^t/K^1)\sin 2\phi]}{(2 + \cos 2\phi)\sin\phi}} \tag{22}$$

i.e. numerically:

$$\varepsilon_T = 4.2\,(\omega - \phi) - 22.2\ K^t/K^1\,, \tag{23}$$

$$\chi_m^2 = -7.9\,(\omega - \phi) + 41.75\ K^t/K^1\,, \tag{24}$$

$$\Delta_E = 0.238\ K^1\,\varepsilon_T^2 = 0.067\ K^1\,\chi_{\mathrm{m}}^4. \tag{25}$$

When only the bond and torsional energies are accounted for *the deformation amplitude depends on 2 parameters only: the torsional to bond constant ratio K^t/K^1 and the angle ω for which the bond angle energy is nil.*

Figure 1 plots K^t/K^1 versus χ_{m} for different ω values.

ΔE (conformational energy difference between planar and deformed ring) is plotted against K^1 for different χ_{m} values in Fig. 2.

These two figures show the great sensitivity of ΔE to χ_{m} and of χ_{m} to ω: the fit of this last parameter thus seems primordial.

If we adopt the value $\chi_{\mathrm{m}} = 47°$ [5] based on the experimental results of McCullough et al. [10] we observe (Fig. 1) that the tetrahedral value $\omega = 109°47$ corresponds to the ratio $K^t/K^1 = 0.0210$; taking the K^t values 3.4 and 2.8 (torsional barriers of propane and ethane) we then obtain K^1 values of 162 and 133, hence (Fig. 2) ΔE values of 4.88 and 4 kcal/mole respectively.

The average value $\omega = 112°7$ found in the n-alkanes and recommended by Lifson [11] leads to K^1 values more consistent with those given by Hendrickson [12] corresponding, still for $\chi_{\mathrm{m}} = 47°$, to the ratio $K^t/K^1 = 0.0316$; thus for $K^t = 3.4$ we find $K^1 = 107.5$ and for $K^t = 2.8$, $K^1 = 88.6$ which correspond in turn to ΔE values of 3.2 and 2.64, respectively.

Generally speaking if we take given conformational characteristics ΔE and χ_{m} we define K^1 (Fig. 2) and at given χ_{m} (Fig. 1), large ω values correspond to large

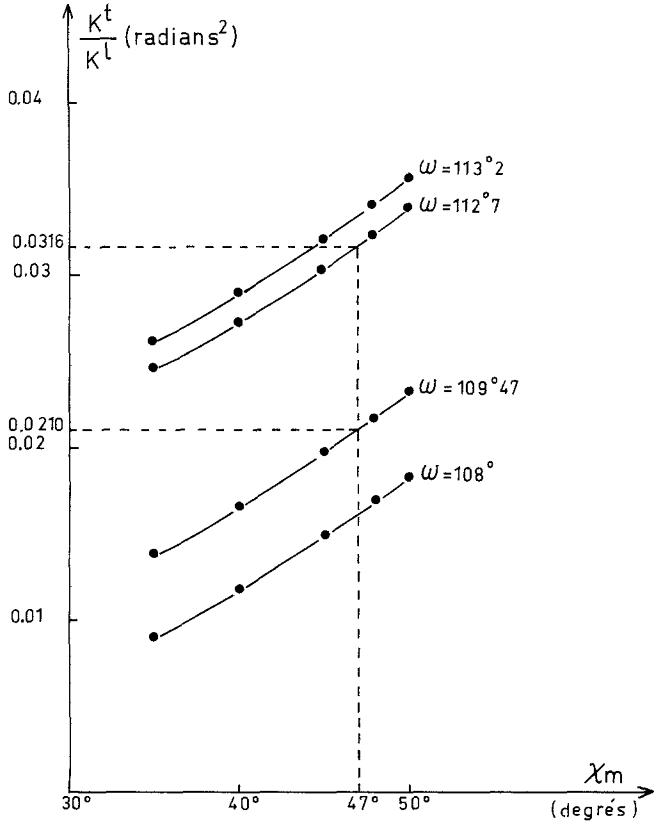

Fig. 1. Variation of the K^t/K^1 ratio versus the χ_m deformation amplitude, for several ω angle values

K^t/K^1 values and hence to large K^t values since K^1 is already fixed. This is probably why for $K^t = 3.4$ kcal/mole the value $\omega = 112°7$ is better suited (values recommended by Lifson [11] whereas for $K^t = 2.8$ the tetrahedral value $\omega = 109°47$ is suitable (values used by Pitzer [13] and Hendrickson [3]).

Let us take as an example the calculations performed by Pitzer [13]: using the parameters $K^1 = 115$, $K^t = 2.8$ and $\omega = 109°47$ he finds χ_m: 48°1 and ΔE (bond + torsion) = 3.64. To arrive at these 2 values of χ_m and ΔE we must have $K^1 = 110$ and, with $\omega = 109.47$, $K^t = 2.4$ and with $\omega = 112°7$, $K^t = 3.56$ (A strict comparison is impossible since no Van der Waals terms have been introduced). We should note that the breakdown of the conformational energy into several terms increases the number of calculation parameters which are implicitly contained in the coefficients A_1 and A_2, and stress the fact that the ring deformation represented by χ_m depends only on the ratio A_1/A_2 while, for a given deformation, amplitude χ_m, ΔE depends only on A_1. This remark is made in the limit of validity of the development (1), and a study is in progress to determine the influence of the introduction of cross-terms between torsion and valence angles.

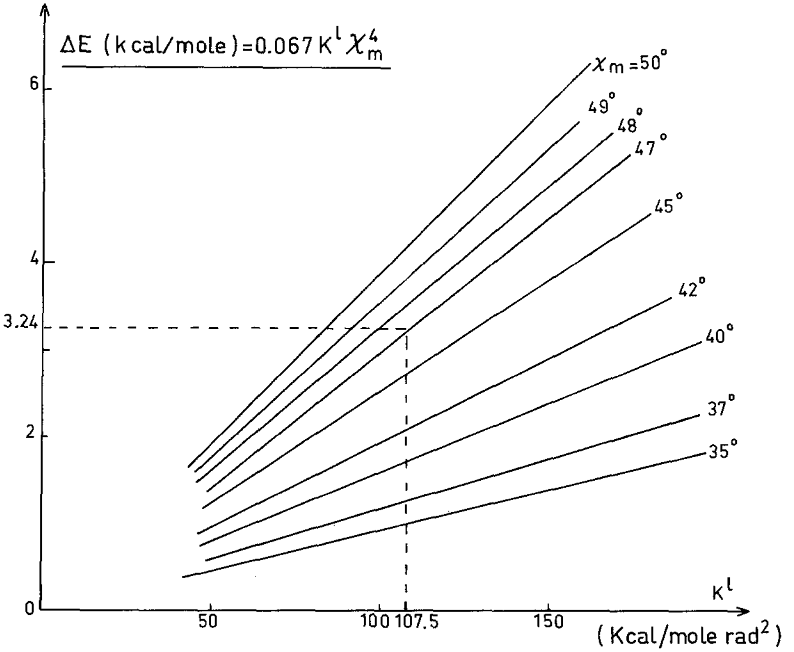

Fig. 2. Energy gain ΔE due to deformation of the ring versus used K^1 value, for several χ_m deformation amplitude

Conclusion

Let us sum up the origins of the conformational properties of the regular 5-membered ring brought out here:

1) the torsional angles χ_i are related by the parametric expressions:

$$\chi_i = \chi_m \cos [P + (i - 1)\delta]$$

for strictly geometric reasons in the case of the C_2 and C_S conformations with equal puckering amplitude; intermediate conformations involve 4 geometrically independent parameters, hence endless geometrically possible paths including the pseudorotational path allow passage from one symmetrical conformation to the next;

2) the deformation amplitude χ_m is related geometrically to the sum of the bond angles ϕ_i by the expression:

$$\chi_m^2 \, sin^2\phi \sin 2\phi = \sum_i (\phi_i - \phi) .$$

This amplitude is therefore defined by either χ_m or $\varepsilon_T = \sum_i (\phi_i - \phi)$.

(This fact was signaled by Dunitz [6]).

3) the bond angles ϕ_i and the deformation amplitude χ_m are geometrically indeterminate;

4) because of the approximate relations:

$$\sum_i \chi_i = 0 \text{ and } k \sum_i (\phi_i - \phi) = \sum_i \chi_i^2 ,$$

the conformational energy may be expressed by the development:

$$E = A_0 + A_1 \sum_i (\phi_i - \phi) + A_2 \sum_i (\phi_i - \phi)^2 ,$$

5) the minimal conformational energy condition $dE = 0$ leads to the definition, as a function of the ratio A_1/A_2:
a) of the deformation amplitude χ_{m} (or ε_T):

$$\chi_m^2 = \alpha\, A_1/A_2 \quad \text{with } \alpha = -5/[2 \sin^2\phi \sin 2\phi\, (2 + \cos 2\phi)] ,$$
$$\varepsilon_T = \beta\, A_1/A_2 \quad \text{with } \beta = -5/[2(2 + \cos 2\phi)] ,$$

b) of the relations between the bond angles ϕ_i; these parametric relationships, like those concerning the torsional angles, are of the pseudorotational type:

$$\phi_i = (\phi + \varepsilon_T/5) + [(2\varepsilon_T \cos 2\phi)/5] \cos \{2[P + (i-1)\delta] + \pi/5\} ,$$

c) in the case of C_S symmetry with $\chi_1 = 0$:

$$\phi_1 = \phi_2 = \phi + \varepsilon_T/10$$
$$\phi_3 = \phi_5 = \phi + \varepsilon_T\,(2 + \cos 2\phi)/5$$
$$\phi_4 = \phi - 2\varepsilon_T\,(\cos 2\phi)/5 ,$$

d) in the case of C_2 symmetry with $\chi_1 = \chi_{\mathrm{m}}$:

$$\phi_1 = \phi_2 = \phi + 3\varepsilon_T/10$$
$$\phi_3 = \phi_5 = \phi - \varepsilon_T(\cos 2\phi)/5$$
$$\phi_4 = \phi + 2\varepsilon_T(1 + \cos 2\phi)/5 ,$$

6) according to the combined conformational energy and bond angle expressions the energy is constant along the pseudorotational circuit. The difference ΔE between the planar and deformed ring energies being:

$$\Delta E = \frac{5A_1^2}{4A_2\,(2 + \cos 2\phi)} = -\frac{A_1}{2}\chi_{\mathrm{m}}^2 \sin^2\phi \sin 2\phi = -\frac{A_1}{2}\varepsilon_T$$

the pathway geometrical indetermination on the non-symmetrical conformations is abolished.
7) Since the conformational energy is considered as the sum of the bond and torsional energies the 2 coefficients A_1 and A_2 are replaced by the 3 parameters K^t, K^1 and ω.
a) The deformation amplitude ε_T varies linearly with ω and K^t/K^1 and depends on these 2 parameters only:

$$\varepsilon_T = b\,[(\omega - \phi) + 9(K^t/K^1)\sin 2\phi] \text{ with } b = 5/(2 + \cos 2\phi)$$

similarly:

$$\chi_{\mathrm{m}}^2 = c\,[(\omega - \phi) + 9(K^t/K^1)\sin 2\phi] \text{ with } c = 5/[(2 + \cos 2\phi) \sin^2\phi \sin 2\phi] .$$

b) The conformational energy difference between planar and deformed ring is proportional to K^1 and χ_m^4: (or K^1 and ε_T^2):

$$\begin{aligned} \Delta E &= d\, K^1 \chi_m^4 \quad \text{with } d = (2 + \cos 2\phi)\sin^2\phi/16 \\ &= e\, K^1 \varepsilon_T^2 \quad \text{with } e = (2 + \cos 2\phi)/5 \,. \end{aligned}$$

e) The use of the following values for the parameters K^1, ω and K^t:

$\omega = 112°7$, $K^t = 3.4$ kcal/mole, $K^1 = 105$ kcal/mole radian^2 leads to the conformational characteristics:

$\chi_m = 48°$ and $\Delta E = 3.5$ kcal/mole.

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Appendix

Notations used

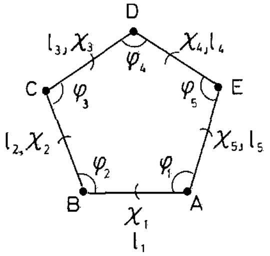

Fig. 3. Geometrical characteristics of the five-membered ring

l_i = bond lengths ≡ 1 in the case of the regular ring
χ_i = torsional angles
ϕ_i = bond angles

ϕ = $3\pi/5 = 108°$
δ = $4\pi/5 = 144°$
ε_i = $\phi_i - \phi$ = complement to the 108° bond angle ϕ_i
ε_T = $\sum_i \varepsilon_i$
E = Conformational energy of the 5-membered ring

ΔE = Conformational energy difference between planar and deformed ring
A_n = Coefficients of the energy development in terms of $\sum_i (\varepsilon_i)^n$
E_i^1 = Bond energy relative to atom i (associated with angle ϕ_i)
K_i^1 = Bond constant relative to atom i (associated with angle ϕ_i)
E_i^t = Torsional energy relative to the i^{th} bond (associated with angle χ_i)
K_i^t = Torsional constant relative to the i^{th} bond (associated with angle χ_i)
ω_i = Angle ϕ_i value corresponding to zero bond energy E_i^1 (when the 5 atoms of the ring are identical, $\omega_i \equiv \omega$)
r_{ij} = Distance between atoms i and j
q_i = Charge carried by atom i
e = Constant of proportionality including the effective dielectric constant
A_{ij}, B_{ij}, d_{ij} = Coefficients dependent on the nature of the atoms i and j and accounted for in the Van der Waals energy and hydrogen bond expressions
$S(r_{ij})$ = Electrostatic contribution to the hydrogen bond energy
P = Pseudorotation phase angle
χ_m = Maximum torsional angle value characterising the deformation amplitude